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# Structural Topology Optimization for the Natural Frequency of a Designated Mode

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The homogenization method and the density function method are common approaches to evaluate the equivalent material properties for design cells composed of matter and void. In this research, using a new topology optimization method based on the homogenized material with a penalty factor and the chessboard prevention strategy, we obtain the optimal layout of a structure for the natural frequency of a designated mode. The volume fraction of nodes of each finite element is chosen as the design variable and a total material usage constraint is imposed. In this paper, the subspace method is used to evaluate the eigenvalue and its corresponding eigenvector of the structure for the designated mode and the recursive quadratic programming algorithm, PLBA algorithm, is used to solve the topology optimization problem.

**Key Words:** Topology Optimization, Equivalent Material Properties, Volume Fraction, Subspace Method, Recursive Quadratic Programming Algorithm

## 1. Introduction

Since the late of 1980s, two types of topology optimization method, i.e. the homogenization method (Bendsφe and Kikuchi, 1988) and the density function method (Yang and Chuang, 1994), utilizing the concept of the design domain are frequently used in topology optimization. The homogenization method places infinitely many microscale rectangular holes in design cells forming perforated materials. Thus, the sizes of the hole, e. g. sides of a rectangular and hole's orientation angle, in each design cell are chosen as the design variables. The number of design variables of this method is very large because the finite element method is required to divide the design domain into many design cells. On the other hand, the density function method uses the density of each design cell as the design variables to formulate the topology optimization problem. This method is very attractive to the engineering community because of its simplicity, but the shortcoming is the lack of theoretical support of the relationship between the density and the material properties (Gea, 1994).

Most topology optimization methods using displacement-based finite element method have in common an undesirable feature that material is distributed in the chessboard patterns. The formation of chessboard patterns is known as a numerical phenomenon similar in nature to the element locking (Bendsφe, Diaz and Kikuchi, 1993). This phenomenon makes the interpretation of the optimal layout very difficult. So, it is necessary to prevent the formation of chessboard patterns. In order to overcome the chessboard problem, the use of higher order elements (Jog, Haber and Bendsφe, 1993), image processing method (Sigmund, 1994), density redistribution algorithm (Park and Youn, 1997a, Youn and Park, 1997b) etc. have been studied in literature. The method using higher order elements has a drawback,

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which means that the size of the problem becomes too large to be practical because a large number of elements are needed to manage the necessary resolution. In optimum iteration process, some methods inspect the formation of chessboard patterns over design domain and reduce the porous region in the optimal density distribution. The methods which use a scheme to prevent the formation of chessboard patterns in optimum iteration process may have effect on the stability of the optimization algorithm.

The natural frequency of a certain vibration mode needs to be increased in order to reduce the displacement of a particular point on the structure (Kawabe and Yoshida, 1994). Kawabe and Yoshida proposed a density distribution method for designing light and rigid structures that maximized the natural frequency of the structure for a certain mode. They used the density of each element as design variable and the Voigt-Reuss model to formulate the relationship between the density and the effective Young's modulus. Based on the density function method, Yang and Chuang (Yang and Chuang, 1994) have studied the optimization of a structure to maximize the lowest eigenvalue. Using the homogenization method and the modified optimality criteria method, Ma, Kikuchi and Hagiwara (Ma et al. 1993) have considered a frequency response optimization problem for both the optimal layout and reinforcement of an elastic structure.

In this research, using a new topology optimization method based on a homogenized material with the penalty factor and a chessboard prevention strategy, we obtain the optimal layout of a structure for the natural frequency of a designated mode. In order to increase the natural frequency of the most important mode by optimizing the shape of the structure, the volume fraction of nodes of each finite element is chosen as the design variable and a total material usage constraint is imposed. We consider a 3-dimensional beam which is subjected to several boundary conditions. And then, we idealize the beam to a plane structure and analyze the optimal material layout of the idealized beam for the natural frequency of a designated mode. Finally, using the optimal material layout of the idealized beam, we assume that the change of volume fraction is equal to the change of thickness of the beam. In that case, the optimal layout and shape of 3-dimensional beam will be obtained. In this paper, the subspace method (Bathe, 1996) is used to evaluate the eigenvalue and its corresponding eigenvector of the structure for the designated mode and the recursive quadratic programming algorithm, PLBA algorithm (Lim, 1985), is used to solve the topology optimization problem. Also, a commercial software, ANSYS package, is used to check up the result frequency of numerical topology optimization examples.

## 2. Homogenized Material and Chess-board Prevention Strategy

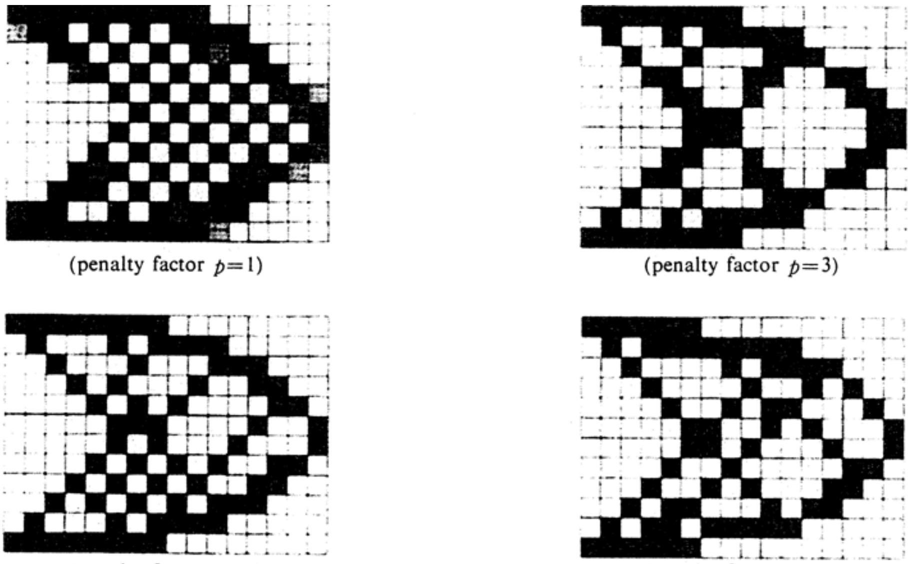
A new microstructure based design domain method was proposed to solve the optimal topology problems (Gea, 1994). Similar to the model of the homogenization method, this method chooses a microstructure with isotropic spherical voids embedded in a matrix. The material properties of the porous medium are derived by means of Mori-Tanaka's mean field theory (Mori and Tanaka, 1973) and Eshelby's equivalence principle (Eshelby, 1957). This method gives simple closed form expressions for equivalent Young's modulus and equivalent shear modulus in terms of the volume fraction of each design cell as follows :

$$\frac{E_H}{E_o}=\frac{c_o}{2-c_o} \tag{1}$$

$$\frac{\mu_H}{\mu_o}=\frac{8c_o}{15-7c_o} \tag{2}$$

In the Eqs. (1) and (2), $E_H$ and $\mu_H$ are the equivalent Young's modulus and the equivalent shear modulus, respectively, $E_o$ and $\mu_o$ are the Young's modulus and the shear modulus of the matrix, respectively, and $c_o$ is the volume fraction of the matrix of each design cell.

In this research, we modify Eq. (1) using a penalty factor because the result of topology optimization based on Eq. (1) has many intermediate design values between lower and upper



(penalty factor $p=1$) (penalty factor $p=3$)

(penalty factor $p=4$) (penalty factor $p=5$)

**Fig. 1** The effects of the penalty factor

bounds (Lim and Lee, 1998). The modification form of Eq. (1) is as follows :

$$\frac{E_H}{E_o}=\frac{c_o^{\,p}}{2-c_o} \tag{3}$$

where, $p$ is the penalty factor which has a value not less than unity.

The effects of the penalty factor on the result of topology optimization of a short cantilever beam are presented in Fig. 1.

As shown in Fig. 1, the result of topology optimization using a penalty factor less than '3' has many intermediate design values between lower and upper bounds and the chessboard patterns are appeared remarkably in this result. The results of topology optimization using a penalty factor more than '3' are represented by the rough optimum layout. In the case of a penalty factor $p=3$, the result of topology optimization has the design values neighboring the lower bounds or the upper bounds and shows the smooth optimum layout. Thus, we select '3' for the penalty factor in this research.

And then, a new strategy is used in this paper to prevent chessboard patterns. This strategy has no effects on the stability of the optimization algorithm. This strategy uses the volume fraction of each node as the design variable. Using the linear shape function in the finite element analysis, the final values of design variables are interpolated over the each element (Lim and Lee, 1999). Using this strategy, Eq. (3) can be modified as follows :

$$\frac{E_H}{E_o}=\frac{\left(\sum_{i=1}^{4}c_iL_i\right)^p}{2-\left(\sum_{i=1}^{4}c_iL_i\right)} \tag{4}$$

In the above, $c_i$ is the volume fraction of each node, $L_i$ is the shape function and '4' is the total number of nodes of each plane-stress element.

## 3. Topology Optimization Problem for Natural Frequency

As shown in Fig. 2, consider a mechanical element as a body occupying a design domain which is chosen so as to allow for a definition of the applied loads and boundary conditions. In this paper, we will solve the optimal topology of the structure with the objective of maximizing the natural frequency of a designated mode, while a mass constraint is imposed to limit material usage. Selecting the natural frequency of a designated mode as the objective function and the volume fraction of each node as the design variable, the topology optimization problem is writ-

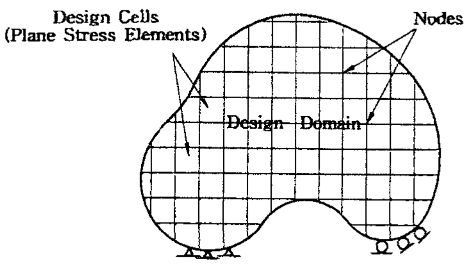



**Fig. 2** A structure for the topology optimization

ten as :

$$\begin{aligned} &\text{Maximize} : f_i(c_j) \\ &\text{Subject to} : K(c_j)\,y=\lambda M(c_j)\,y \\ &\qquad m_{total}(c_j)-m_o\leq 0 \\ &\qquad 0< c_{\min.}\leq c_j\leq c_{\max.},\ j=1\cdots n. \end{aligned} \quad (5)$$

In the above, $f_i$ is the natural frequency of a designated mode, $K$ and $M$ are the global stiffness matrix and the global mass matrix, $\lambda$ and $y$ are the eigenvalue and its corresponding eigenvector, respectively, $m_{total}$ is the total mass and $m_o$ is the mass limit of the structure. Also, $c_j$ is the volume fraction of each node, $c_{\min.}$ and $c_{\max.}$ are the lower and upper bounds, respectively, and $n$ is the total number of design variables which is equivalent to the total number of nodes.

As topology defined by the volume fraction of each node varies, the structural properties must vary accordingly. This implies that the material property coefficients defined in the matrix $K$ and $M$ must depend on the volume fraction. Thus, the stress-strain relation matrix $[D_e]$ for a two-dimensional plane stress case and the equivalent density $\rho_H$ may be written as :

$$[D_e]=\frac{\left(\sum_{j=1}^{4}c_jL_j\right)^p}{2-\left(\sum_{j=1}^{4}c_jL_j\right)}\frac{E_o}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} \quad (6)$$

$$\rho_H=\rho_o\sum_{j=1}^{4}c_jL_j \quad (7)$$

where, $\nu$ is the Poisson's ratio and $\rho_o$ is the density of the matrix.

## 4. Sensitivity Analysis

The recursive quadratic programming algorithm, PLBA algorithm, uses the derivatives of the objective function and the constraint function with respect to design variables to search for a optimum direction. Using finite element models, the calculation of design sensitivity analysis can be performed by two different computational schemes : the analytical method and the finite differencing approximation. The analytical method for sensitivity analysis has two different methods : the direct differentiation method and the adjoint variable method. The direct differentiation method of the analytical method is used here because the relations between the equivalent properties and design variables are explicitly derived from the results in the present study.

### 4.1 The sensitivity analysis of the objective function

The natural frequency $f_i$ of the designated mode $i$ is related to the eigenvalue $\lambda_i$, as shown in the following equation :

$$f_i=\frac{\sqrt{\lambda_i}}{2\pi} \quad (8)$$

In this research, to evaluate the sensitivity of the natural frequency, we evaluate the sensitivity of the eigenvalue. And then, using the following equation, the sensitivity of the natural frequency can be evaluated,

$$\frac{\delta f_i}{\delta c_j}=\frac{1}{4\pi\sqrt{\lambda_i}}\frac{1\,\delta\lambda_i}{\delta c_j} \quad (9)$$

where, $c_j$ denotes the design variable.

Premultiplying Eq. $K(c_j)\,y=\lambda M(c_j)\,y$ of Eqs. (5) by the transpose of the eigenvector $y^T$, the following equation is obtained :

$$y^TK(c_j)\,y=\lambda y^TM(c_j)\,y \quad (10)$$

We approximate both sides of Eq. (10) to first order variation in the variables $y$, $\lambda$, and $c_j$. And then, using the normalized eigenvector $y$ with $y^TM(c_j)\,y=1$, the following sensitivity equation of the eigenvalue with respect to the design variable is obtained :

$$\frac{\delta\lambda_i}{\delta c_j}=y_i^T\frac{\partial K(c_k)}{\partial c_j}y_i-\lambda_i y_i^T\frac{\partial M(c_k)}{\partial c_j}y_i \quad (11)$$

If the objective function of Eq. (5) is only simple, non-repeated frequency, the design sensitivity expression of the objective function is Eq. (11). However, it is recognized that the design sensitivity analysis of non-simple, repeated eigenvalues is difficult due to the fact that they generally are not continuously differentiable, but are only directionally differentiable. Accordingly, design sensitivities for non-simple repeated eignvalues cannot be determined by expression such as Eq. (11), unless special precautions are taken. For more information about the design sensitivity analysis of non-simple, repeated eigenvalues, refer to Ref. (Haug et al. 1986).

### 4.2 The sensitivity analysis of the constraint function

Eq. $m_{total}(c_j) - m_o \leq 0$ of Eq. (5) can be expressed as follows :

$$g = \sum_{i=1}^{N} \rho_i a_i \left( \sum_{j=1}^{4} L_j c_j \right) - m_o \leq 0. \tag{12}$$

In the above, using Eq. (7), $\rho_i$ can be calculated, $a_i$ is the area of each design cell, and $N$ is the total number of design cells.

As shown in Eq. (12), the constraint function is linear in the design variable. Thus, the sensitivity analysis of the constraint function can be directly evaluated with respect to the design variable $c_j$ as follows :

$$\frac{\delta g}{\delta c_j} = \sum_{i=1}^{N_i} a_i L_i, \tag{13}$$

where, $N_i$ is the total number of neighboring design cells of the $j$ th node.

## 5. Numerical Examples

In this section, we consider the optimal topology of a structure with the objective of maximizing the natural frequency of a designated mode, while a mass constraint is imposed to limit material usage. All the examples have $E_o = 207$ $Gpa$, $\rho_o = 7700 Kg/m^3$, and $\nu = 0.3$. Initially, the nodal volume fraction values are set equal to unity at each node so that the geometry is identical to the design domain which do not has holes. And then, the lower and upper bounds of the design variable are set to '$10^{-3}$' and '1', respectively.

### 5.1 Clamped beam

Consider a structure, the height is $1m$, the thickness is $1m$, and the length is $7m$, subjected to its middle left and right sides are clamped. And then, this structure is idealized as a plane structure which is shown in Fig. 3.

For the finite element analysis, $32 \times 8$ four-node elements are used. Thus, this model consists of 256 elements and 297 nodes (design variables).

#### 5.1.1 The bending mode about z-axis

In this example, we consider the optimal topology of the clamped beam with the objective of maximizing the natural frequency of the bending mode about z-axis, while the quantity of the material to be used is 40% of the design domain. Figure 4 shows the result obtained from the new topology optimization method.

As shown in Fig. 4, the white design cells represent the voids. The more black a design cell is, the larger a thickness is. Thus, the most black design cells represent that the thickness of the design cells is $1m$. Figure 5 shows the 3-dimensional result of the optimal topology for the bending mode.

Figure 6 shows the result mode of the optimal structure. The ANSYS package is used to check up the result frequency and designated mode of

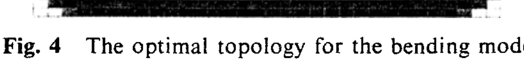



**Fig. 3** Clamped beam at middle points of the left and right sides

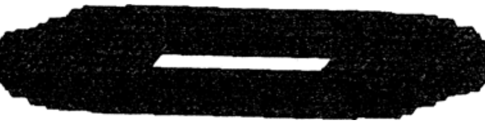


**Fig. 4** The optimal topology for the bending mode

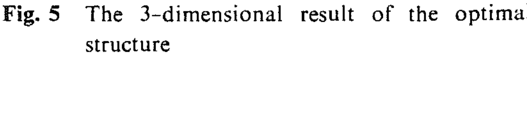


**Fig. 5** The 3-dimensional result of the optimal structure

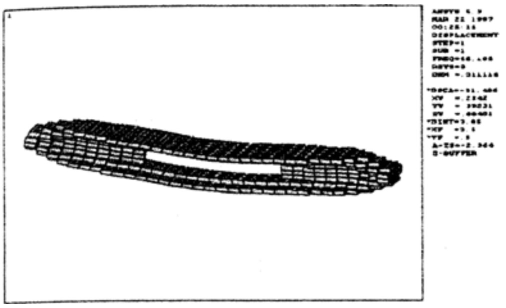


**Fig. 6** The bending mode of the optimal structure

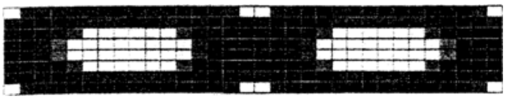


**Fig. 7** The optimal topology for the axisymmetric mode

**Fig. 8** The 3-dimensional result of the optimal structure

numerical topology optimization examples. As shown in Fig. 6, the natural frequency of the initial structure were 46.36*Hz* and the result one of the optimal structure is 56.19*Hz*.

#### 5.1.2 The axisymmetric mode about z-axis

In this example, we consider the optimal topology of the clamped beam with the objective of maximizing the natural frequency of the axisymmetric mode about z-axis, while the quantity of the material to be used is 60% of the design domain. Figure 7 shows the result obtained from the topology optimization method used in this research.

Figure 8 shows the 3-dimensional result of the optimal topology for the axisymmetric mode.

And then, Fig. 9 shows the result mode of the optimal structure. The natural frequency of the initial structure were 169.78*Hz* and the result one of the optimal structure is 203.95*Hz*.

### 5.2 Fix-supported beam

As shown in Fig. 10, a fix-supported beam at

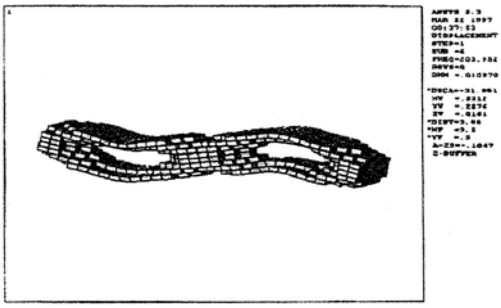


**Fig. 9** The axisymmetric mode of the optimal structure

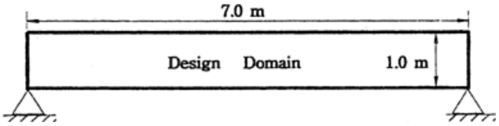



**Fig. 10** Fix-supported beam at the end points

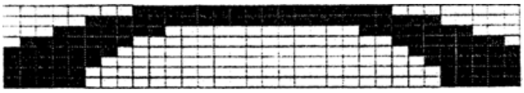


**Fig. 11** The optimal topology for the bending mode

**Fig. 12** The 3-dimensional result of the optimal structure

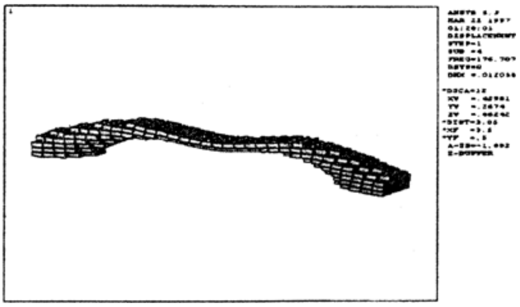


**Fig. 13** The bending mode of the optimal structure

the end points of the bottom edge is considered. And then, we consider the optimal topology of the fix-supported beam with maximizing the natural frequency of the bending mode about z-axis. The total material usage constraint is 40% of the design domain. For the finite element analysis, $32 \times 8$ four-node elements are used. Thus, this model consists of 256 elements and 297 nodes.

Figure 11 shows the optimal topology of the fix-supported beam from the new topology optimization method.

Figure 12 shows the 3-dimensional result of the optimal topology for the bending mode. And then, Fig. 13 shows the result mode of the optimal structure. The natural frequency of the initial structure were $75.84 Hz$ and the result one of the optimal structure is $176.71 Hz$.

## 6. Conclusion

The natural frequency of a certain vibration mode needs to be increased in order to reduce the displacement of a particular point on the structure. In this research, using the homogenized material with the penalty factor and the chessboard prevention strategy, a topology and shape optimization method is developed to increase the natural frequency of the certain vibration mode of a structure. General mathematical programming tools can then be formulated and solved using this method. As shown in the numerical examples, use of this method makes it possible to obtain the optimal structure, which is not only optimal in size and shape but also in topology. It is expected that this method will provide a new and simple alternative for solving topology optimization problem of a vibrating structure.

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